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Review

Nonlinear stability of an axial electric field: Effect of interfacial charge relaxation

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A R T I C L E I N F O

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A B S T R A C T

We introduce a nonlinear perturbation technique to third order, to study the stability between two cylindrical inviscid fluids, subjected to an axial electric field. The study takes into account the relaxation of electrical charges at the interface between the two fluids. At first order, a linear dispersion relation is obtained. Analytical and numerical results for the overstability and incipient instability conditions are given. For perfect dielectric fluids, the electric field has a stabilizing influence, while for leaky dielectric fluids, the electric field can have either a stabilizing or a destabilizing influence depending on the conductivity and permittivity ratios of the two fluids. At higher order, a nonlinear dispersion relation (nonlinear Ginzburg–Landau equation) is derived, describing the evolution of wave packets of the problem. For leaky dielectric fluids near the marginal state, a nonlinear diffusion equation (nonlinear incipient instability) is obtained. For perfect dielectric fluids, two cubic nonlinear Schrödinger equations are obtained. One of these equations to determine a nonlinear cutoff electric field separating stable and unstable disturbance, whereas the other is used to analyze the stability of the system. It is found that the nonlinear stability criterion depending on the ratio of permittivity, Such effects can only be explained successfully in the nonlinear sense, as the linear analysis unsuccessful to inform about them.

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1. Introduction

The interaction of electric fields and free or polarization charges with moving fluids and their common interface, makes electrohydrodynamic fluid flow a very complicated phenomenon. Electrohydrodynamic instability of the interface between two fluids stressed by an axial electric field has received considerable attention [1–3]. The stability in the presence of an axial electric field in cylindrical perfect dielectric fluids for axisymmetric case, was first reported by Nayyar and Murty [1]. They found that the uniform axial electric field has a strong stabilizing influence on the cylindrical interface for short and long wavelengths in all symmetric and axisymmetric modes of perturbation, where the two fluids (cylinder and surroundings) are treated as perfect dielectrics. Some of these works have assumed a perfect conductor, so that charge relaxation can be considered instantaneous see in Saville [2]. Elhefnawy et al. [3] studied the stability of a finitely conducting (leaky dielectrics) compound jets in an axial electric field. They found that the uniform axial electric field has a stabilizing or destabilizing influence, according to some of conditions on conductivity and permittivity ratios.

In perfect dielectrics case, there is an electrostatics force directed normal to an interface between two isotropic dielectrics due to the difference in dielectric constants, while the shearing force has no implication since the interface is free of charge [4]. On the other hand, conductors may be have just such a surface charge, and hence an electrical shearing force in addition to the normal force. This shearing force is identically zero, if the electric field is directed along the axis of a cylinder and remains so until the cylindrical interface is deformed. The study of the polarization charges has attracted the attention of many investigators [5–9]. Melcher and Schwarz [5] studied the effect of a tangential electric field on a planar interface separating two imperfect fluids, in the present of charge relaxation. Saville [6] demonstrated that the charge relaxation instabilities of [5] can occur for perfect dielectrics, in contrast to Melcher and Schwarz [5]. Further information and references as well as an excellent review can be found at Melcher and Taylor [7]. Mestel [8] investigated the effects of surface charge, axial field and a finite charge relaxation for a slight Reynolds number flow. Chen et al. [9] discussed the small and large electrical relaxation time limit of a viscous coflowing jet in a radial electric field. They concluded that the accuracy of the dispersion relation approximation.

The problem of nonlinear hydrodynamic analysis has been investigated by several authors [10–12]. They showed that some effects of physical problems can be explained more precisely in nonlinear sense, when the linear theory fails to predict them.

The equations of electrohydrodynamic flow are containing nonlinear terms. Therefore, the phenomenon cannot be discussed entirely by a linear theory (see for example [13–17]). Elhefnawy et al. [13–15] investigated the nonlinear stability of cylindrical structures of finitely conducting fluids influenced by electric field in different situations. In their analysis, two nonlinear Schrödinger and Klein–Gordon equations are derived. They obtained new instability regions, as the stability diagrams for different sets of physical parameters. Elcoot [16] investigated the nonlinear electroviscous potential flow analysis. He sowed that the nonlinear stability condition depend on the viscosity coefficients, which does not explain in the linear theory of viscous potential flow. Elcoot [17] examined the effects of the surface charge and charge relaxation on the nonlinear stability of cylindrical interface by considering various limiting cases in axisymmetric and nonaxisymmetric modes. He introduced solutions for some partial differential equations (nonlinear modified Schrödinger equations) to study the nonlinear stability conditions. Electrohydrodynamics have many industrial applications including electrospraying [18], ink-jet printers [19], as well as nanotechnology [20].

In this paper, we extend our previous works [3] to include the effect of the relaxation of free charges at the interface and convection current. These extensions are interesting mathematically. This paper is organized as follows. In Section 2, we present the mathematical formulation of our electrohydrodynamic problem. In Section 3, we obtain the linear analytical dispersion relation describe the stability of the system. Also, the breakup phenomena of liquid jets into drops are discuss in the light of linear theory. In Section 4, we introduce an analysis based on the Fourier transform and multiple scales method to derive Ginzburg–Landau equation, which represents the nonlinear analytical dispersion relation of our problem. Generally, the nonlinear equations describes the competition between nonlinearity and linear dispersion relation. In Section 5, we discuss some special cases and obtain nonlinear analytical dispersion relations (nonlinear diffusion and Schrödinger equations). In Section 6, we present results of our numerical calculations for the nonlinearized electrohydrodynamic problem, and compare the results with the linear instability theory. Finally draw our conclusions in Section 7. Here, the calculation is greatly complicated in practice because of the intricate algebra, much of which has to be carried out with the help of computer algebra package mathematica.

2. The physical and mathematical model

The considered system is specifically associated with the stability problems of liquid jet of radius R. A parallel flow of two inviscid fluids in infinite, fully saturated, uniform and homogeneous media. The two fluids are incompressible and have uniform properties. The interface between the two fluids is assumed to be well defined and initially circular cylinder. Gravity forces are assumed to be ignored. The cylindrical jet has a surface tension T at the surface of the undeformed jet. The cylindrical polar coordinates (r, θ, z) are considered. In the equilibrium state, the z-axis is the axis of symmetry of the system. The inner and outer fluids have densities ρ_1 and ρ_2, permittivities ε_1 and ε_2 and conductivities σ_1 and σ_2. Generally, the subscripts 1 and 2 refer to the constant physical parameters of the inner and outer fluids, respectively. It is also assumed that the jet is acted upon by the influence of a uniform axial electric field E_0 along the axis of the jet.

The location of the interface is described by

$$F(r,\theta,z,t) = r - R - \eta(\theta,z,t), \tag{2.1}$$

where $F = 0$ represents the bounding surface, $t > 0$ is the time and $\eta = \eta(\theta,z,t)$ is the perturbation in the radius of the interface from its equilibrium value R. The outward normal unit vector $\mathbf{n}$ is written as

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left(1, -r^{-1}\frac{\partial \eta}{\partial \theta}, -\frac{\partial \eta}{\partial z}\right)\left[1 + r^{-2}\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2\right]^{-\frac{1}{2}}. \tag{2.2}$$

2.1. The governing equations

Since large currents are not present in this flow, the effects of magnetic inductions are negligible. Thus, Maxwell's equations [21]:

$$\nabla \wedge \mathbf{E} = 0, \tag{2.3}$$
$$\nabla \cdot \varepsilon \mathbf{E} = q, \tag{2.4}$$
$$\nabla \cdot \mathbf{J} + \frac{\partial q}{\partial t} = 0, \tag{2.5}$$
$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{v} q, \tag{2.6}$$

where q corresponds to the scalar free charge density and $\mathbf{E}$, $\mathbf{J}$ and $\mathbf{v}$ are the vectors of the electric field, the free current density and the fluid velocity, respectively.The parameter ε is the permittivity and the parameter σ is the electrical conductivity of the fluid, which are constants.

The conservation of momentum and mass for the flow are given by

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right] = -\nabla p + \mathbf{F}, \tag{2.7}$$
$$\nabla \cdot \mathbf{v} = 0, \tag{2.8}$$

where p is the pressure and $\mathbf{F}$ is the electric force density vector is $\mathbf{F} = q\mathbf{E}$. The Maxwell equations lead to an exponential decay of the bulk charge density as $\exp(\sigma_1 t/\varepsilon_1)$ where the parameter $\sigma_1 t/\varepsilon_1$ is sufficiently short, so that the electric charge density in the bulk is essentially zero. Therefore, the bulk forces of electrical origin are negligible and the field coupling occurs at the interface as specified by the appropriate boundary conditions [7]. Therefore, the field coupling occurs at the interface as specified by the appropriate boundary conditions.

We assume that the electric field in both fluids can be derived from a potential function $\Phi^{(j)}(r,\theta,z,t)$ such that

$$\mathbf{E}^{(j)} = \nabla\left(E_0 z - \Phi^{(j)}\right), \quad j = 1, 2, \tag{2.9}$$

where the superscripts 1 and 2 refer to the dependant variables of physical quantities of the inner and outer fluids, respectively. The free charge density is initially zero everywhere within each fluid, it will remains zero. It follows from (2.4) and (2.9) that the electrostatic potential satisfies the Laplace's equation:

$$\nabla^2 \Phi^{(j)} = 0, \quad j = 1, 2, \tag{2.10}$$

where, $\nabla \Phi^{(j)} \to 0$ as $|z| \to \infty$. These will be obtained later of solutions periodic in z. These means that the physical quantities must tend to zero.

The velocity can be expressed via the potential function $\Psi^{(j)}(r,\theta,z,t)$ as $\mathbf{v}_j = \nabla \Psi^{(j)}$ and the basic equation governing the perturbed velocity potential is

$$\nabla^2 \Psi^{(j)} = 0, \quad j = 1, 2, \tag{2.11}$$

where, $\nabla \Psi^{(j)} \to 0$ as $|z| \to \infty$.

2.2. Boundary conditions

In addition to, the requirement that all physical quantities must tend to zero, as $r \to 0$ for $j = 1$, and $r \to \infty$ for $j = 2$, we must also impose interfacial boundary conditions.

(i) The kinematic condition requires that the normal velocity of both fluids must equal the velocity of the interface, i.e.,

$$\frac{\partial \Psi^{(j)}}{\partial r} = \frac{\partial \eta}{\partial t} + \frac{1}{R^2}\left(1 + \frac{\eta}{R}\right)^{-2}\frac{\partial \eta}{\partial \theta}\frac{\partial \Psi^{(j)}}{\partial \theta} + \frac{\partial \eta}{\partial z}\frac{\partial \Psi^{(j)}}{\partial z}. \tag{2.12}$$

Since both fluid move together with the interface, and since there is no slip between the fluids in the direction of flow, both the normal and the tangential velocities are continuous. The continuity of the tangential velocity are $|[\Psi_z]| = 0$ and $|[\Psi_\theta]| = 0$, where $|[]|$ represents the difference in a quantity as we cross the interface, i.e., $|[X]| = X_2 - X_1$.

(ii) The tangential component of the electric field is continuous at the interface [17]:

$$|[E_t]| = 0, \tag{2.13}$$

where $E_t(= \mathbf{n} \wedge \mathbf{E})$ is the tangential component of the electric field.

(iii) Conservation of electric charge on the interface [8,17] requires that

$$|[\sigma E_n]| + \frac{DQ}{Dt} - Q\mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{v}] = 0, \tag{2.14}$$

$$Q = |[\varepsilon E_n]|, \tag{2.15}$$

where $E_n(= \mathbf{n} \cdot \mathbf{E})$ is the normal component of the electric field and Q is the surface charge density.

(iv) The continuity of the normal stress yields

$$\left|\left[\rho \frac{\partial \Psi}{\partial t}\right]\right| + \frac{1}{2}|[\rho(\nabla\Psi)^2]| + T\left(\frac{1}{\Re_1} + \frac{1}{\Re_2}\right) = -\frac{1}{2}|[\varepsilon E_n^2]| - \frac{1}{2}|[\varepsilon E_t^2]|, \tag{2.16}$$

where $\Re_1$ and $\Re_2$ are the principal radii of the surface curvature [22]. Now, we will proceed to derive the linear and nonlinear dispersion relations.

3. The linear dispersion relation

The goal here is to be obtained analytical dispersion relation described the stability of the system in the linear problem posed by the Eqs. (2.10)–(2.16). In dealing with the above set of equations, first we expand the various perturbed quantities in the following asymptotic series [23]

$$\eta(\theta, z, t) = \sum_{n=1}^{N+1} \eta_0^n \eta_n(\theta, z, t) + O(\eta_0^{N+2}), \tag{3.1}$$

$$f(\theta, r, z, t) = \sum_{n=1}^{N+1} \eta_0^n f_n(\theta, r, z, t) + O(\eta_0^{N+2}), \tag{3.2}$$

where f is any of the variables $\Phi^{(j)}$, $\Psi^{(j)}$ and η_0 is a small parameter representing the strength of the nonlinearity (i.e. $\eta_0 \ll 1$). To circumvent the problem, we need to expand the boundary conditions (2.12)–(2.16) around $r = R$ by using a Taylor's series. Expressions (3.1) and (3.2) are substituted into Eqs. (2.10) and (2.11), and the transformed boundary conditions, then the coefficients of terms of equal powers in η_0 are equated, we obtain three sets of equations [23].

The linear stability of electric jets of both fundamental and practical interest. On the practical side, the breakup phenomenon of a jet into droplets due to instability has become a key technology, for example internal combustion engines, ink-jet printers, spray coating. On the fundamental side, the stability mechanisms are influenced by for example the surface tension of the interfaces, the densities of jet and surrounding fluid and the electric properties of the fluids. The aim here is analyzes the stability of the problem at hand throughout a linear approach. The linear curve leads to understanding the mechanism of the jet breakup. That is the formation of the main (parent) drops, which occurs in the unstable region.

The first order solution of the linear problem with respect to the z, t and θ is obtained as

$$\eta_1 = A(\theta, z, t)e^{i\phi} + c.c., \tag{3.3}$$

$$\Psi_1^{(j)} = iA_1^{(j)}A(\theta, z, t)e^{i\phi} + c.c., \tag{3.4}$$

$$\Phi_1^{(j)} = iB_1^{(j)}A(\theta, z, t)e^{i\phi} + c.c., \tag{3.5}$$

and

$$\phi = kz + m\theta - \omega t, \quad i = \sqrt{-1}, \quad j = 1, 2, \tag{3.6}$$

where, $c.c.$ denotes the complex conjugate of the preceding terms, $A(\theta, z, t)$ is an unknown varying function denoting the amplitude of the propagating wave will be determined later. This amplitude have the wavenumbers k and m and frequency ω, where k and m are real and positive, but m must be integral and

$$A_1^{(1)} = -\omega \frac{I_m(kr)}{kI'_m(kR)}, \tag{3.7}$$

$$A_1^{(2)} = -\omega \frac{K_m(kr)}{kK'_m(kR)}, \tag{3.8}$$

$$B_1^{(1)} = -\frac{E_0 I_m(kr)K_m(kR)[(\sigma_1 - \sigma_2) - i(\varepsilon_1 - \varepsilon_2)\omega]}{\Sigma(k) - i\mathscr{E}(k)\omega}, \tag{3.9}$$

$$B_1^{(2)} = -\frac{E_0 K_m(kr) I_m(kR)[(\sigma_1-\sigma_2) - i(\varepsilon_1-\varepsilon_2)\omega]}{\Sigma(k) - i\mathscr{E}(k)\omega}, \tag{3.10}$$

$$\Sigma(k) = \sigma_1 K_m(kR) I'_m(kR) - \sigma_2 I_m(kR) K'_m(kR), \tag{3.11}$$

$$\mathscr{E}(k) = \varepsilon_1 K_m(kR) I'_m(kR) - \varepsilon_2 I_m(kR) K'_m(kR). \tag{3.12}$$

In Eqs. (3.7)–(3.12), $I_m(kR)$ and $K_m(kR)$ are the modified Bessel functions of order m, and the prime on the modified Bessel functions denotes differentiation with respect to r at $r = R$. The above mentioned solutions are valid provided that ω, k, m and E_0 satisfied the linear characteristic function

$$D(\omega, m, k, E_0) = \frac{T}{R^2}(1 - m^2 - k^2R^2) + \frac{\rho_1 \omega^2 I_m(kR)}{k I'_m(kR)} - \frac{\rho_2 \omega^2 K_m(kR)}{k K'_m(kR)} - kE_0^2 I_m(kR) K_m(kR) \times \frac{\left[(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2) - i(\varepsilon_1-\varepsilon_2)^2\omega\right]}{\Sigma(k) - i\mathscr{E}(k)\omega} = 0. \tag{3.13}$$

Conclusions drawn directly from (3.13).

If we consider the case of finite conductivity, in which the charge relaxation time can be negligibly small regardless of the fluid. It is mathematically, assumed that $\tau \to 0$. To obtain appropriate form in this case, it is convenient to rewrite the last term of Eq. (3.13) in the form

$$\frac{\left[(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2) - i(\varepsilon_1-\varepsilon_2)^2\omega\right]}{\Sigma(k) - i\mathscr{E}(k)\omega} = \frac{\left[(\varepsilon_1-\varepsilon_2)\left(\frac{\sigma_1}{\sigma_2}-1\right) - i\varepsilon_2\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2(\tau_2^*\omega)\right]}{D_2},$$

where

$$D_2 = \left[\frac{\sigma_1}{\sigma_2} K_m(kR) I'_m(kR) - I_m(kR) K'_m(kR)\right] - i\left[\frac{\varepsilon_1}{\varepsilon_2} K_m(kR) I'_m(kR) - I_m(kR) K'_m(kR)\right](\tau_2^*\omega),$$

and $\tau_2^* = \varepsilon_2/\sigma_2$. Physically, the electric charge relaxation time in the fluid is short compared with the surface dynamics time scale ($\tau_2^*\omega \ll 1$). Thus, the characteristic function (3.13) yields

$$D(\omega, m, k, E_0) = \frac{T}{R^2}(1 - m^2 - k^2R^2) + \frac{\rho_1 \omega^2 I_m(kR)}{k I'_m(kR)} - \frac{\rho_2 \omega^2 K_m(kR)}{k K'_m(kR)} - kE_0^2 I_m(kR) K_m(kR) \times \frac{[(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2)]}{\Sigma(k)} = 0. \tag{3.14}$$

When, $m = 0$ is considerable, then the characteristic function (3.14) gives Elhefnawy et al. [3].

If we consider the case, $\sigma_1 = 0$ and $\sigma_2 = 0$, then the characteristic function (3.13) reduces to

$$\frac{T}{R^2}(1 - m^2 - k^2R^2) + \frac{\rho_1 \omega^2 I_m(kR)}{k I'_m(kR)} - \frac{\rho_2 \omega^2 K_m(kR)}{k K'_m(kR)} - \frac{kE_0^2(\varepsilon_1-\varepsilon_2)^2 I_m(kR) K_m(kR)}{\mathscr{E}(k)} = 0. \tag{3.15}$$

This form appropriate for perfect dielectrics. Nonaxisymmetric deformations ($m \neq 0$) are always stable; axisymmetric deformations with wavenumbers within the range $0 < kR < 1$ are unstable [6]. The formula for the axisymmetric case ($m = 0$) was first reported by Nayyar and Murty [1].

The linear characteristic function (3.13) gives the linear dispersion relation

$$(-i\omega)^3 + a_2(-i\omega)^2 + a_1(-i\omega) + a_0 = 0, \tag{3.16}$$

which is a cubic equation in ω with complex coefficients and

$$a_2 = \frac{\Sigma(k)}{\mathscr{E}(k)}, \tag{3.17}$$

$$a_1 = \frac{k I'_m(kR) K'_m(kR)}{\varrho(k)}\left[\frac{T}{R^2}(k^2R^2 + m^2 - 1) + k I_m(kR) K_m(kR) \times \frac{E_0^2(\varepsilon_1-\varepsilon_2)^2}{\mathscr{E}(k)}\right], \tag{3.18}$$

$$a_0 = \frac{k I'_m(kR) K'_m(kR) \Sigma(k)}{\varrho(k)\mathscr{E}(k)}\left[\frac{T}{R^2}(k^2R^2 + m^2 - 1) + k I_m(kR) K_m(kR) \times \frac{E_0^2(\varepsilon_1-\varepsilon_2)(\sigma_1-\sigma_2)}{\Sigma(k)}\right], \tag{3.19}$$

$$\varrho(k) = \rho_1 I'_m(kR) K_m(kR) - \rho_2 I_m(kR) K'_m(kR). \tag{3.20}$$

It can easily be shown that all the roots (or their real parts) of Eq. (3.16) are negative if and only if $a_0, a_1, a_2 > 0$ [24] (and $a_1 a_2 > a_0$, if Eq. (3.16) has complex roots). Therefore, the system is linearly stable if

$$a_1 a_2 > a_0 \to (\varepsilon_1-\varepsilon_2)(\varepsilon_1\sigma_2 - \varepsilon_2\sigma_1) > 0, \tag{3.21}$$

$$a_0, > 0 \rightarrow T(k^2R^2 + m^2 - 1) + \frac{kR^2 I_m(kR)K_m(kR)E_0^2(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)}{\Sigma(k)} > 0, \tag{3.22}$$

$$a_1, > 0 \rightarrow T(k^2R^2 + m^2 - 1) + \frac{kR^2 I_m(kR)K_m(kR)E_0^2(\varepsilon_1 - \varepsilon_2)^2}{\mathscr{E}(k)} > 0. \tag{3.23}$$

We now consider the possibility that the marginal state are characterized static instability. Therefore, as $\omega \rightarrow 0$ in Eq. (3.13), we obtain the following condition for the incipient instability (the principal of exchange of stability) becomes

$$\frac{T}{R^2}(k^2R^2 + m^2 - 1) + kE_{c1}^2 I_m(kR)K_m(kR) \times \frac{[(\varepsilon_1 - \varepsilon_2)(\sigma_1 - \sigma_2)]}{\Sigma(k)} = 0, \tag{3.24}$$

where E_{c1} is the critical electric field separates stable regions from unstable regions. It is apparent from the relation (3.24) reduces to Rayleigh [25] in the absence of electric field for the hydrodynamic jet whose maximum growth of instability is at $k = 0.678$ for $R = 1$ and the critical wavenumber occurs when $k = 1$. It is clear that the relation (3.24) depends on the sign of both $(\varepsilon_1 - \varepsilon_2)$ and $(\sigma_1 - \sigma_2)$. This result was demonstrated by many authors for axisymmetric mode in electrohydrodynamics [3,26].

Before dealing with the numerical calculations in the linear theory, must be considering the linear dispersion relation (3.16) in an appropriate dimensionless form, which consistent with the physical measure. This can be written in the dimensionless form

$$\tau_2\omega_0^3 + b_2\omega_0^2 + \tau_2 b_1\omega_0 + b_0 = 0, \tag{3.25}$$

with

$$\begin{aligned}
b_2 &= \frac{f(\sigma)}{f(\varepsilon)},\\
b_1 &= \frac{\kappa I'_m(\kappa)K'_m(\kappa)}{f(\rho)}\left[\kappa^2 + m^2 - 1 + \kappa E\frac{I_m(\kappa)K_m(\kappa)(\varepsilon - 1)^2}{f(\varepsilon)}\right],\\
b_0 &= \frac{\kappa I'_m(\kappa)K'_m(\kappa)f(\sigma)}{f(\rho)f(\varepsilon)}\left[\kappa^2 + m^2 - 1 + \kappa E\frac{I_m(\kappa)K_m(\kappa)(\varepsilon - 1)(\sigma - 1)}{f(\sigma)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&f(\mathfrak{X}) = \mathfrak{X} I'_m(\kappa)K_m(\kappa) - I_m(\kappa)K'_m(\kappa), \qquad \mathfrak{X} = \rho, \varepsilon, \sigma,\\
&\kappa = kR, \quad \sigma = \frac{\sigma_1}{\sigma_2}, \quad \varepsilon = \frac{\varepsilon_1}{\varepsilon_2}, \quad \rho = \frac{\rho_1}{\rho_2},\\
&\omega_0 = (-i\omega)\left(\frac{\rho_2 R^3}{T}\right)^{1/2}, \quad \tau_2 = \frac{\varepsilon_2}{\sigma_2}\left(\frac{T}{\rho_2 R^3}\right)^{1/2}, \quad E = \frac{\varepsilon_2 R E_0^2}{T}.
\end{aligned}$$

Therefore, the corresponding linear stability conditions are

$$(\varepsilon - 1)(\varepsilon - \sigma) > 0, \tag{3.26}$$

$$(\kappa^2 + m^2 - 1) + \frac{\kappa I_m(\kappa)K_m(\kappa)E(\varepsilon - 1)(\sigma - 1)}{f(\sigma)} > 0, \tag{3.27}$$

$$(\kappa^2 + m^2 - 1) + \frac{\kappa I_m(\kappa)K_m(\kappa)E(\varepsilon - 1)^2}{f(\varepsilon)} > 0. \tag{3.28}$$

For leaky dielectric fluids corresponding to $\tau_2 \rightarrow 0$ and mode $m = 0$, we found that the stability condition depends only on the relation (3.27). This means that the electric field has a stabilizing effect depending on $\kappa > 1$ and $(\varepsilon - 1)(\sigma - 1) > 0$. Therefore, for leaky dielectrics, the electric field can have either a destabilizing or a stabilizing effect depending on the ratios of permittivities and conductivities of the two fluids. While, for perfect dielectrics as $m = 0$ is considerable, then the electric field has a stabilizing effect (Nayyar and Murty [1] with different notation).

In what follows, we shall make a numerical estimation for the stability picture of surface waves propagating between two cylindrical fluids. In order to screen this examination, numerical calculations for the dispersion relation (3.25) and the relations (3.26)–(3.28) are made for the variation of the dimensionless growth rate ω_0 and the electric field E intensity versus the wavenumber. In the following figures the stable region is referred by S, while U stands for the unstable one.

We will discuss the oscillatory instability (overstability), which corresponding to the real part of ω_0 of the linear dispersion relation (3.25). Numerical results obtained from the dispersion relation (3.25) to determine the growth rate at different wavenumbers for various initial conditions and fluid properties, for the axisymmetric mode $m = 0$ and for some values of $\sigma$are shown in Fig. 1. It is observed that the mode of maximum instability decreases with increase $\sigma(= \sigma_1/\sigma_2)$. This means that the effect of conductivity ratio $\sigma$is stabilizing with the increase of the wavenumber κ, when $(\varepsilon - 1)(\sigma - 1) > 0$. While in

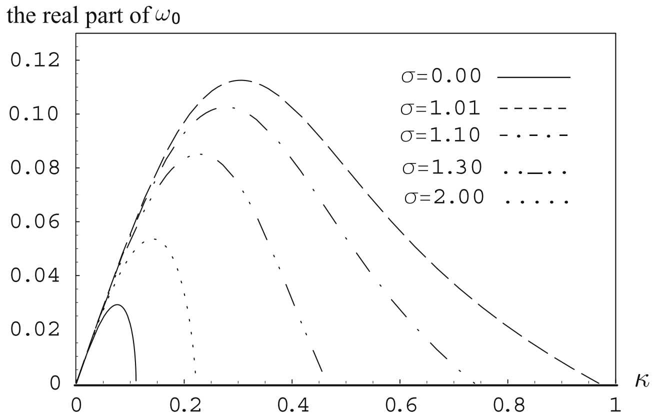

Fig. 1. Effect of different values of the parameter σ on the growth rate and critical wavenumber for linear system. The input parameters are $m = 0$, $\rho = 1.5$, $\varepsilon = 4$, $\tau_2 = 1$ and $E = 4$. The graph indicates the relation (3.25) for different values of $\sigma > 1$.

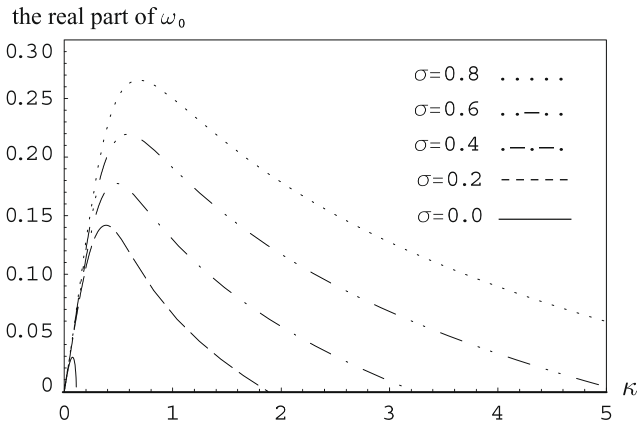

Fig. 2. Stability diagram for the same system as in Fig. 1, but for different values of $\sigma < 1$.

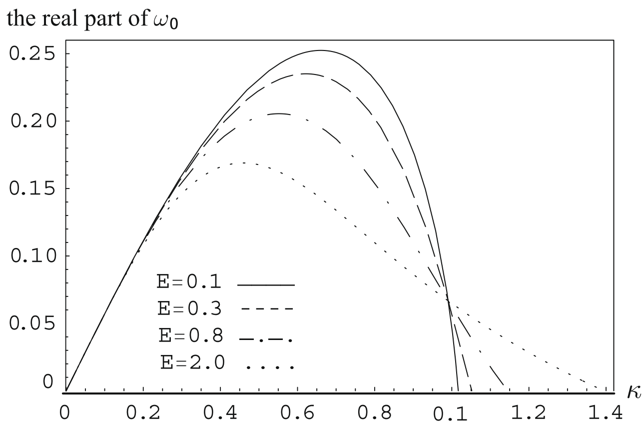

Fig. 3. Stability diagram for the same system as in Fig. 2, but with $\sigma = 0.8$, for different values of E.

Fig. 2, we observe that the linear system has a destabilizing effect (the curves change their behaviour) with increase σ, as $(\varepsilon - 1)(\sigma - 1) < 0$. Which means that the ratios of permittivities and conductivities of the two fluids play a dual role in the stability analysis. In the stability diagram given in Fig. 3, we also examine the dispersion relation (3.25) as $m = 0$ and $(\varepsilon - 1)(\sigma - 1) < 0$ for different values of E. As shown in the Fig. 3, the position at which the $(\omega_0 - k)$-curves belonging to

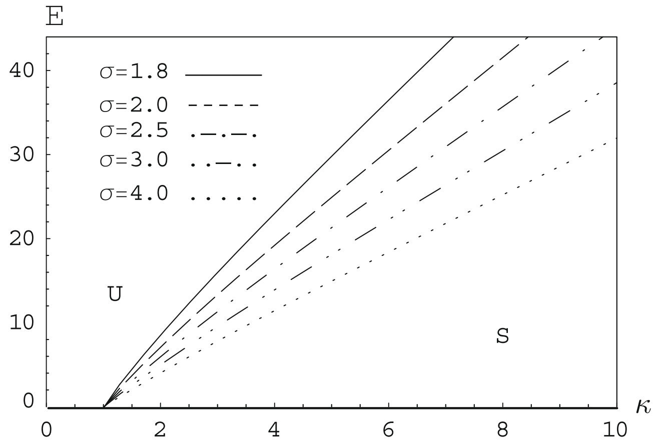

Fig. 4. Stability diagram in the $(E - \kappa)$-plane, for linear system having $m = 0$, $\varepsilon = 0.5$ and $\tau_2 = 1$. The graph indicates the relation (3.27) for different values of $\sigma > 1$.

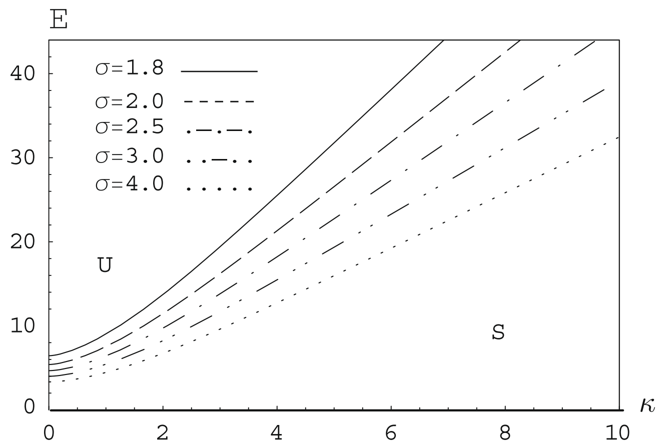

Fig. 5. Stability diagram for the same system as in Fig. 4, but with $m = 1$.

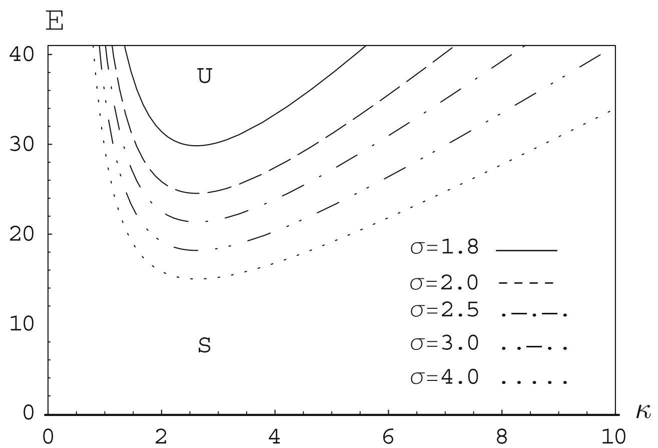

Fig. 6. Stability diagram for the same system as in Fig. 4, but with $m = 2$.

different values of E cross each other is noteworthy. By reading the curves in Fig. 3, we can deduce the value of κ for different values of E. It is found that this position occurs for value of $\kappa = 0.989$, and the electric field has a destabilizing effect for $\kappa > 0.989$. While all unstable modes are within the range $0 < \kappa < 0.989$. It is also found that the progressive shifting of

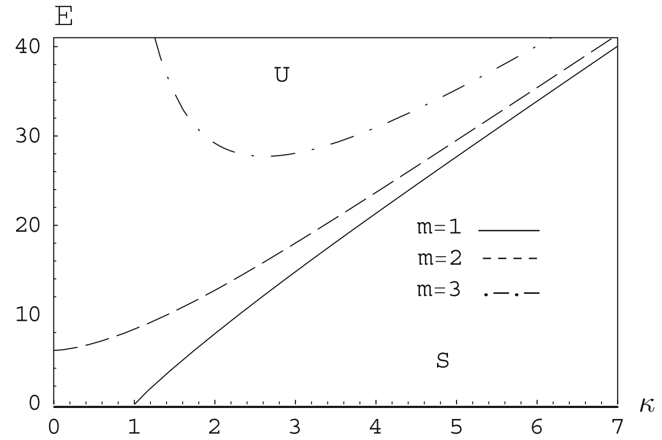

Fig. 7. Stability diagram for the same system as in Fig. 4, but with $\sigma = 2$, for different values of m.

the mode of maximum instability towards smaller values of κ as E increases is apparent. Thus, for a large sufficient value of E, the cylinder will not have a tendency to breakup. By comparing the curves in Fig. 3, we can determine the value of κ corresponding to the value of mode of maximum. This mode depending on electric field, liquid properties and cylinder radius, from which we determine the drop size [27]. Therefore, under the influence of an electric field, a liquid drop surrounded by another liquid changes its shape. The breakup phenomena of the jet into drops is playing an important role in practical applications [28].

In Figs. 4–7, we shall study the principle of exchange of stabilities $((\varepsilon - 1)(\sigma - 1) < 0)$ for special modes of $m = 0$, $m = 1$ and $m = 2$. In Figs. 4–6, we shall examine the stability for different values of conductivity ratio σ on the field. While the Fig. 7, is used to comparing the stability on the field with respect to the modes. Thus, the limit of Eq. (3.25) as $\omega \to 0$ represents the condition for incipient instability as in (3.27). We consider the modes of $m = 0$, $m = 1$ and $m = 2$, for Figs. 4–6, respectively. As can be seen in these figures, the electric field has a destabilizing effect in this case with respect to the conductivity σ. This means that the electric field has a destabilizing effect with respect to the inner conductivity σ_1, while, this field has a stabilizing effect with respect to the outer conductivity σ_2. In Fig. 7, we investigate the influence of the modes on the stability of the field. By comparing the curves in Fig. 7. It is found that the modes of cylinder play an important role in the linearity for incipient instability, where the modes have a stabilizing effect on the system.

4. The nonlinear dispersion relation

Since our aim is to study the amplitude modulation of traveling waves or wavepackets which can exist when both dispersion and weak nonlinearity are present. In the analysis of the behaviour of such waves the concept of the nonlinear dispersion relation, where the frequency depends both on the wavenumber of the carrier wave and on the amplitude. Such a concept is useful for the physicist; it brings out clearly the origins of the dispersive and the nonlinear terms in the slow evolution of the wave envelope compared to that of the rapid carrier wave. Therefore, we carry out the problem in the dimensional form of the second order set of equations, one may substitute the solutions of the first order problem into the second order ones and solve the resulting equations. Thus, the second harmonic term η_2 is given by

$$\eta_2 = \Omega A^2 \exp[2i\phi] + c.c., \tag{4.1}$$

where $\Omega(= \alpha/D(2\omega, 2m, 2k, E_0))$, and α are under request from the author. Note that the nonzero dominator $D(2\omega, 2m, 2k, E_0)$ may be derived from the linear dispersion relation Eq. (3.13) by replacing ω, m, k by $2\omega, 2m2k$, respectively. The vanishing of the dominator $D(2\omega, 2m, 2k, E_0)$ refers to the second harmonic resonance. In general, harmonic resonance may occur if (ω, m, k) and $(n\omega, nm, nk)$ satisfy the same dispersion relation [29]. When resonance occurs, we find that both the surface distribution and the excited volume pulsation undergo modulation. Experiment, the interaction occurred between harmonics of the disturbance. It predicts the existence of satellites for all wavenumbers. Here, we have assumed that $D(2\omega, 2m, 2k, E_0) \neq 0$.

The third order problem may obtain it, when we substitute the first and second order solutions into the three order ones and with vanishing of all coefficients of the factor implied the nonlinear characteristic function

$$D(\omega, m, k, E_0, |A|^2) = D(\omega, m, k, E_0) - \eta_0^2 G|A|^2 = 0, \tag{4.2}$$

where the nonlinear coefficient G which may be positive or negative. It is describe the behaviour of the physical parameters of the system in the nonlinear sense, which is lengthy and not included here. Here G depend on R, T, k, m, ω, ρ_1, ρ_2, ε_1, ε_2, σ_1, σ_2 and E_0. It is available from the author on request and outlined by [10].

To investigate the stability of nonlinear characteristic function (4.2), we may use an expansion procedure based on the method of multiple scale [29]. The underlying idea of the method of multiple scales is make an expansion representing the solution of the problem not only as a function of one independent variable, but also as a function of two or more independent variables which are referred to as scales. The independent variables z and t, which are measured on scale of a typical wavelength and periodic time, may be extended to introduce alternative independent variables

$$Z_n = \eta_0^n z, \quad T_n = \eta_0^n t, \quad n = 0, 1, 2. \tag{4.3}$$

Thus, defining Z_0, T_0 as variables appropriate to fast variations for linear case, corresponding to k_0, ω_0, respectively. Here k_0, ω_0 represent the wavenumber and the angular frequency for the dimensional linear case, respectively. Z_1, Z_2, T_1 and T_2 are slow variables, with considerable the mode m is constant [30]. Using Eq. (4.3) and the chain rule, hence the time and space derivatives become

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \eta_0 \frac{\partial}{\partial T_1} + \eta_0^2 \frac{\partial}{\partial T_2} + \cdots \tag{4.4}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z_0} + \eta_0 \frac{\partial}{\partial Z_1} + \eta_0^2 \frac{\partial}{\partial Z_2} + \cdots \tag{4.5}$$

Therefore, we can be rewritten η in the form

$$\eta = A(z,t) e^{i(k_0 z + m\theta - \omega_0 t)} + c.c. \tag{4.6}$$

Here, we have introduced the idea of a control parameter which controls the system externally, for example E_0. The term proportional to A is identified in the above with a change of E_0. This procedure is relevant for example when a change in causes the onset of instability and the subsequent build up of the amplitude [17,31]. Assuming $E_0 = E_c \pm \eta_0^2$, where E_0 is the applied electric field, E_c is the critical electric field. This parameter represents the departure of the system from the bifurcation value E_c. Thus, we discuss here the stability of the system in the neighborhood of the neutral curve by setting the bifurcation parameter E_0 as $E_c \pm \eta_0^2$.

We can expand $D(\omega, m, k, E_0, |A|^2)$ as a function of ω, k, E_0 and $|A|^2$ in a Taylor series about the wavenumber k_0, the angular frequency ω_0, the constant amplitude A_0, and E_c. We obtain expansions of powers of $\delta\omega(= \omega - \omega_0)$ and $\delta k(= k - k_0)$, where $\delta\omega$ and δk are small. With neglect, the high order of $\delta\omega^2$ and δk^2. Therefore

$$D + \frac{\partial D}{\partial \omega}\delta\omega + \frac{\partial D}{\partial k}\delta k + \frac{1}{2}\left[\frac{\partial^2 D}{\partial \omega^2}\delta\omega^2 + 2\frac{\partial^2 D}{\partial\omega\partial k}\delta\omega\delta k + \frac{\partial^2 D}{\partial k^2}\delta k^2\right] = \eta_0^2\left(\pm\frac{\partial D}{\partial E_0} + G|A|^2\right). \tag{4.7}$$

The nonlinear coefficient G evaluated at $A = A_0 = 0$, is given by $G = (\partial D/\partial |A|^2)|_{A_0}$. The relation (4.7) is the dispersion relations of the envelope wave. This dispersion relation evaluate in the vicinity of ω_0, related k_0 and A_0. We can obtain a general equation describes the evolution of the envelope function $A(z,t)$ for waves (4.6). It may be derived by considering the Fourier transform for the envelope wave as

$$A(\delta\omega, \delta k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(z,t) e^{i(\delta\omega t - \delta k z)}\, dz\, dt. \tag{4.8}$$

Therefore, the inverse transform of (4.8) can be written in the form

$$A(z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\delta\omega, \delta k) e^{-i(\delta\omega t - \delta k z)}\, d\delta\omega\, d\delta k. \tag{4.9}$$

We have

$$\frac{\partial A}{\partial t} = -i\delta\omega A, \quad \frac{\partial A}{\partial z} = i\delta k A. \tag{4.10}$$

Multiplying Eq. (4.7) by $A(z,t)$ and insert Eq. (4.10) into (4.7), afterwards using the Eqs. (4.4) and (4.5), we obtain expansions of powers of η_0, where $\delta\omega$ and δk are small, i.e., of order η_0. Equating coefficients of like powers of η, we get the following set for D:

At the first order, we obtain the same result as in (3.13), i.e., $D = 0$. While the second and third order problems, gives the following system of partial differential equations

$$-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial T_1} + \frac{\partial D}{\partial k}\frac{\partial A}{\partial Z_1} = 0. \tag{4.11}$$

$$i\left(-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial T_2} + \frac{\partial D}{\partial k}\frac{\partial A}{\partial Z_2}\right) + \frac{1}{2}\frac{\partial^2 D}{\partial \omega^2}\frac{\partial^2 A}{\partial T_1^2} - \frac{\partial D}{\partial\omega\partial k}\frac{\partial^2 A}{\partial Z_1 \partial T_1} + \frac{1}{2}\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial Z_1^2} = \left(\pm\frac{\partial D}{\partial E_0} + G|A|^2\right)A. \tag{4.12}$$

By manipulations, similar arguments as given by Elhefnawy et al. [13], one finds the nonlinear partial differential equations, which defined the nonlinear Ginzburg–Landau equation

$$i\frac{\partial A}{\partial \zeta} + (P_r + iP_i)\frac{\partial^2 A}{\partial \xi^2} = (Q_r + iQ_i)|A|^2 A + (R_r + iR_i)A, \tag{4.13}$$

where

$$Q_r + iQ_i = -G\left(\frac{\partial D}{\partial \omega}\right)^{-1},$$
$$2(P_r + iP_i) = \frac{d^2\omega}{dk^2} = \frac{dv_g}{dk},$$
$$(R_r + iR_i) = \left(\pm\frac{\partial D}{\partial E_0}\right)\left(\frac{\partial D}{\partial \omega}\right)^{-1},$$
$$\xi = \eta_0(z - v_g t) \quad \text{and} \quad \zeta = \eta_0^2 t,$$

the physical quantity $v_g = d\omega/dk = -(\partial D/\partial k)(\partial D/\partial \omega)^{-1}$ is the group velocity of the wave train and the coefficient $(P_r + iP_i)$ represents the group velocity dispersion. The real parts P_r, Q_r and R_r does not depend on the conductivities σ_1 and σ_2, while the imaginary parts P_i, Q_i and R_i are functions of σ_1 and σ_2. The Ginzburg–Landau equation (4.13) is dispersion relation in the nonlinearity. It may be used to study the stability behaviour of the considered system. Lange and Newell [32] derived the stability criteria of this equation. If the solution of Eq. (4.13) is linearly perturbed, the perturbations are stable if both the following conditions

$$P_rQ_r + P_iQ_i > 0 \quad \text{and} \quad Q_i < 0, \tag{4.14}$$

provided that $R_r = 0$. Otherwise, the system is unstable (i.e., the system does not oscillate about its equilibrium state). The transition curves separating the stable from the unstable region correspond to

$$Q_i = 0, \tag{4.15}$$
$$P_rQ_r + P_iQ_i = 0. \tag{4.16}$$

These marginal curves may be born out of numerical estimation.

5. Special cases

The Ginzburg–Landau equation (4.13) is nonlinear dispersion relation describing the evolution of wave packets. This equation gives the nonlinear stability conditions (4.14). We shall study the Ginzburg–Landau equation under some special cases in the light of physical phenomenons. This study gives equations leads to define new nonlinear unstable regions, which tell about satellite drops may be to occur in these regions. Donnelly and Glaberson [33] showed experimentally that the main drops are interspersed with smaller drops (satellites), which could not be accounted for by linear theory. The first theoretical attempt to be extended the linearized asymptotic model of Rayleigh [25] to include higher order corrections for the nonlinear terms was by Yuen [34]. He used the method of strained coordinates. Yuen's theory showed that the interaction occurred between harmonics of the disturbance. It predicts the existence of satellites for all wavenumbers which Goedde and Yuen [35] discussed in their experimental results. Subsequently, Nayfeh [36] applied the method of multiple scales to the same problem that Yuen treated and showed that this method leads to erroneous results near the cutoff wavenumber. Malik and Singh [37] developed a solution for electrohydrodynamic jet. They showed that the satellite are always present for all values of wavenumbers, when an external electric field is applied to the jet.

The newly generated disturbances or the higher harmonics may be stable if their wavenumber is larger than the cutoff wavenumber. Therefore, a better standing of the behaviour of electric jet subject to disturbances with wavenumbers larger than the cutoff wavenumber (i.e., stable region) can help in explaining the influence of the higher harmonics on the jet instability and in designing disturbances to better control the jet breakup process [38].

The next step in the analysis of jet stability determines the stable and unstable region in the nonlinear theory, for example the behaviour of an electric field in the parameter space.

5.1. Nonlinear diffusion equation

We first discuss the case of the nonlinear principle of exchange of stability. We notice that the condition $R_r = 0$ is satisfied when incipient instability occurs, i.e., $\omega = 0$, therefore $P_r = Q_r = 0$. In this case, Eq. (4.13) reduces to the nonlinear diffusion equation [39]

$$\frac{\partial A}{\partial \zeta} + P_i\frac{\partial^2 A}{\partial \xi^2} = Q_i|A|^2A + R_iA. \tag{5.1}$$

The solution of the nonlinear diffusion Eq. (5.1) is valid near the marginal state in the case $\omega = 0$ and can, therefore be used to study the stability of the system. From the inequalities (4.15) we find that the stability conditions of Eq. (5.1) are

$$P_i < 0 \quad \text{and} \quad Q_i < 0. \tag{5.2}$$

The stability can therefore be discussed by dividing the $E_0^2 - \sigma$ plane into stable and unstable regions. The transition curves are given by the vanishing of $P_i = 0$ and $Q_i = 0$.

5.2. Nonlinear Schrödinger equation

A special case occurs when the electrical conductivity is negligible by taking $\sigma_1 = 0$ and $\sigma_2 = 0$ in the evolution equation (4.13). Therefore, in Eq. (4.13) both P_i, Q_i and R_i are equal to zero. Therefore, Eq. (4.13) is reduced to the nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial \zeta} + P_r\frac{\partial^2 A}{\partial \xi^2} = Q_r|A|^2A + R_rA, \tag{5.3}$$

where P_r, Q_r and R_r are the real parts of P, Q and R respectively, when $\sigma_1 = 0$ and $\sigma_2 = 0$. Removing the linear term R_rA by an appropriate phase shift [40]. The solutions of the cubic Schrödinger equation (5.3) has been extensively studied by [10,30,41]. From the theory of the nonlinear Schrödinger equation, Eq. (5.3) is completely integrable and has soliton solutions. The analytic form for a single-soliton solution is given by

$$A = v_1\left[1 - \left\{\left[1-(v_1^2/v_2^2)\right]sn^2\left[\sqrt{(-Q_r/2P_r)}(\xi - u_e\zeta)\right]\right\}\right]^{\frac{-1}{2}}\exp\left[i\frac{u_e}{2P}(\xi - u_c\zeta)\right],$$

where v_1 and v_2 are arbitrary functions determined by the initial conditions, u_e is the envelope velocity, u_c is the carrier velocity and sn is the Jacobian elliptic function. Therefore, we find that the electrohydrodynamic wave is modulationally stable or unstable according to whether $P_rQ_r > 0$ or $P_rQ_r < 0$, respectively. In general the signs of both P_r and Q_r play an important role to determine the stability and instability regions. Thus, the marginal curve is

$$P_rQ_r = 0, \tag{5.4}$$

this equation gives two transition curves

$$P_r = 0, \quad \text{and} \quad Q_r = 0, \tag{5.5}$$

which separating the stable from the unstable region. The solution of (5.3) also break down as denominator of Ω tends to zero. Physically, it represents the second harmonic resonance. Therefore, the nonlinear transition curves are given by the vanishing the group velocity rate P_r and the nonlinear interaction parameter Q_r as will as the real second harmonic resonance curve $D_r(2\omega, 2m, 2k, E_0) = 0$. The phenomenon of resonance arise because of the occurrence of zero divisors in the nonlinear interaction parameter Q_r. We observe the curve of the second harmonic resonance is dependent of the electric field. These curves are transition curves.

5.3. Nonlinear cutoff electric field (perfect dielectric)

It is observed that the solution of Eq. (5.3) is not valid at $\partial D_r/\partial\omega = 0$ (i.e., $\omega \to 0$) where

$$D_r = D|_{\substack{\sigma_1=0\\ \sigma_2=0}} \quad \text{and} \quad G_r = G|_{\substack{\sigma_1=0\\ \sigma_2=0,}}$$

the carrier wave is a standing wave rather a progressive wave. For this case, the group velocity rate P_r and the nonlinear interaction parameter Q_r become singular as $\omega \to 0$. The previous analysis, therefore needs to modify.

We now discuss the case when $\omega \to 0$, corresponds to the electric field in the neighbourhood of the linear cutoff electric field. Thus, we shall study the solvability conditions (4.11) and (4.12) when the frequency near zero, for $\sigma_1 = 0$ and $\sigma_2 = 0$.

In the case of the perfect dielectric fluids, the solvability conditions (4.11) and (4.12) can be simplified and combined together to produce a single equation in the form:

$$\left(\frac{\partial A}{\partial z} + \frac{dk}{d\omega}\frac{\partial A}{\partial t}\right) + \frac{i}{2}\frac{d^2k}{d\omega^2}\frac{\partial^2 A}{\partial t^2} = -i\eta_0^2[G_r/(\partial D_r/\partial k)]|A|^2A, \tag{5.6}$$

where $\partial D_r/\partial k \neq 0$ and $dk/d\omega = -(\partial D_r/\partial\omega)/(\partial D_r/\partial k)$ is the inverse of the group velocity for perfect dielectrics.

Eq. (5.6) can easily be transformed into a nonlinear Schrödinger equation by means of the Gardner–Morikawa transformation [3]

$$\mathrm{Z} = \eta_0^2 z, \quad \mathrm{T} = \eta_0\left(t - \frac{dk}{d\omega}z\right), \tag{5.7}$$

Eq. (5.6) is then become

$$i\frac{\partial A}{\partial \mathrm{Z}} - \frac{1}{2}\frac{d^2k}{d\omega^2}\frac{\partial^2 A}{\partial \mathrm{T}^2} = [G_r/(\partial D_r/\partial k)]A^2\overline{A}. \tag{5.8}$$

Eq. (5.6) admits the following solution for temporal variation [3]:

$$A = \frac{1}{2}A_0\exp[ist + const.], \tag{5.9}$$

where A_0 is a constant, and s is given by

$$s = \left\{\frac{dk}{d\omega} \pm \left[\left(\frac{dk}{d\omega}\right)^2 - \frac{\eta_0^2A_0^2}{2(\partial D_r/\partial k)}\frac{d^2k}{d\omega^2}G_r\right]^{1/2}\right\}\bigg/\frac{d^2k}{d\omega^2}. \tag{5.10}$$

The square root can be expanded in powers of η_0. Thus

$$s = \frac{\eta_0^2 A_0^2 G_r}{4(\partial D_r/\partial\omega)}. \tag{5.11}$$

To get a valid expansion near $(\partial D_r/\partial\omega) = 0$, i.e., in the limit as $\omega \to 0$, we get

$$\frac{dk}{d\omega} = \omega\frac{d^2k}{d\omega^2}. \tag{5.12}$$

From Eqs. (5.10) and (5.12), we obtain

$$s = \omega - \left[\varkappa(E_0) - \frac{1}{4}\eta_0^2 A_0^2 G_r\right]^{1/2}, \tag{5.13}$$

where

$$\varkappa(E_0) = \frac{kI'_m(kR)K'_m(kR)}{\varrho(k)}\left[\frac{T}{R^2}(1-m^2-k^2R^2) + kI_m(kR)K_m(kR)\times\frac{E_0^2(\varepsilon_1-\varepsilon_2)^2}{\mathscr{E}(k)}\right]. \tag{5.14}$$

The cutoff wavenumber can be obtained from Eq. (5.13) by equating the root to zero. This leads to

$$\varkappa(E_0) = \frac{1}{4}\eta_0^2 A_0^2 G_r, \quad \text{at } E_0 = E_{c2}. \tag{5.15}$$

The cutoff electric field can be obtained by solving the above Eq. (5.15). If we let

$$E_0 = E_{c2} + \eta_0^2 E_2 + O(\eta_0^4), \tag{5.16}$$

into Eq. (5.15), we get for the zero order in η_0

$$\frac{T}{R^2}(1-m^2-k^2R^2) + kI_m(kR)K_m(kR)\frac{E_0^2(\varepsilon_1-\varepsilon_2)^2}{\mathscr{E}(k)} = 0, \tag{5.17}$$

which is the linear case. For the order of η_0^2 we get

$$E_2 = -\frac{A_0^2}{2\varkappa'(E_0)}G_r|_{E_0=E_c}, \tag{5.18}$$

where the prime on $\varkappa(E_0)$ denotes differentiation with respect to E_0. The nonlinear cutoff electric field, receive a second order increment $\eta_0^2E_2$. The nonlinear effect is stabilizing if E_2 is negative and vice versa, which depends on the equilibrium radial field.

6. Numerical results

Because of the complexity of the nonlinear dispersion relation, a simplification of the dimensionless form is adopted here to overcome this complexity. This simplification is focused to facilitate the procedure of the nonlinear numerical calculations. It is convenient to write the stability conditions in an appropriate dimensionless form. This can be do in a number of ways depending primarily on the choice of the characteristic length. Consider the following dimensionless forms: the characteristic length $= R$ (the radius of the undisturbed jet), the characteristic time $= 1/\omega(\omega$ is the frequency of the disturbance), and characteristic mass $= \rho_2 R^3$. The corresponding dimensionless quantities are given

$$k = k^*/R, \quad T = T^*\rho_2R^3\omega^2, \quad \sigma_j = \sigma_j^*/\omega \quad E_0 = E_0^*\omega R(\rho_2\varepsilon_2)^{\frac{1}{2}}, \tag{6.1}$$

where superposed asterisks refer to dimensionless quantities. From now, it will be omitted for simplicity. To this end, the interface of the system becomes stable or unstable depending on whether the electric field intensity E_0 is large or smaller than E_c. In general, the polarization electric effect E_c represents, the critical electric field, separates stable regions from unstable regions. It is coupling of two cases. One of which is $E_c = E_c^{con.}$ for the leaky dielectrics fluids [3], while the other is $E_c = E_c^{ins.}$ for perfect dielectric fluids (Nayyar and Murty [1] in the axisymmetric mode with different notation) where

$$E_c^{con.} = \left[T(1-m^2-k^2)(\sigma K_m(k)I'_m(k) - I_m(k)K'_m(k))\right]^{1/2} \times [kI_m(k)K_m(k)(\varepsilon-1)(\sigma-1)]^{-1/2}, \tag{6.2}$$

$$E_c^{ins.} = \left[T(1-m^2-k^2)(\varepsilon K_m(k)I'_m(k) - I_m(k)K'_m(k))\right]^{1/2} \times (\varepsilon-1)^{-1}(kI_m(k)K_m(k))^{-1/2}. \tag{6.3}$$

Generally, we shall deal with the nonlinear analysis in the dimensionless form (6.1). The analysis of the jet instability is determine the stable and unstable regions through the nonlinear theory, for example the behaviour of the electric field in the parameter space. Now we shall study the behaviour of the linear stable region of the system through the nonlinear theory. Therefore, the nonlinear transition curves will divide the linear stable region into stable and new nonlinear unstable parts. Thus, the stability can be discussed by dividing the plane of the stability diagrams into stable region symbolized as S and unstable region symbolized as U. While, the nonlinear curves produce newly shaded unstable regions U_1 and U_2. The comparison between the linear and nonlinear theory make the stability of system more accurately.

The linear analysis play an important role to determine the unstable region, which lies in it the main drops. While, through the nonlinear analysis appears new unstable regions, which lies in it the satellite drops. In the following figures, we intend to determine new nonlinear unstable regions, which inform about satellite drops may be to occur in these regions. The topic of satellite drop volume in electrohydrodynamic is discussed in more detail in reference [37]. In what follows, the analysis of jet stability determines the stable and unstable region in the nonlinear theory, for example the behaviour of an electric field in the parameter space.

In Fig. 8, we shall discuss numerically, the general case (Ginzburg–Landau equation (4.13)) of the relaxation charge through inviscid fluids. The analysis of the system depends on the linear conditions (6.2) and (6.3) and the nonlinear conditions (4.15) and (4.16) in the dimensionless sense (6.1). After lengthy but straightforward calculations, the transition curves $Q_i = 0$ and $P_rQ_r + P_iQ_i = 0$, may be rearranged in two polynomials in E_0^2. We also observe that the resonance curve has no implication on the stability criterion in this case, because of the denominator conjugate occurrence of the complex P and Q always makes the denominator positive. The linear stability condition (6.2) depend on the values of σ_1 and σ_2. Therefore, the stability can be discussed by dividing $(E_0^2 - T)$-plane into stable S (above the curve) and unstable region U (below the curve). While, the nonlinear curves produce newly shaded unstable regions U_1 and U_2, which are confined between the transition curves $Q_i = 0$ and $P_rQ_r + P_iQ_i = 0$ as in Fig. 8. In this figure, the continued line represents the linear curves representing the relations (6.2) and (6.3). While the dotted lines indicate the nonlinear curve $P_rQ_r + P_iQ_i = 0$ and the dashed dotted lines represent the nonlinear curve $Q_i = 0$. Through Fig. 8, we shall consider the linear and the nonlinear stability criteria for some sample chosen system at value of the wavenumber $k = 0.1$ and $k = 0.2$. In the light of the linear theory, the linear curve (6.2) do not appear in this figure because it lie in the negative part of E_0^2-axis. While, we notice that the linear curve (6.3) shifts downwards with the increase of k, which means that the stable region S increases, while the unstable region U decreases when values of k are increasing. It follows that the electric field becomes stable in the light of linear theory with increasing k. Through the nonlinear approach, it is found that the new unstable regions U_1 and U_2 decrease as k increases. The inspection of the nonlinear transition curves shows that the nonlinear effects partition the stable region into stable and new unstable parts. It found that the electric field has a stabilizing effect in the light of nonlinear theory with increasing k. This means that the electric field has a stabilizing effect in the linear and the nonlinear theory with respect to the wavenumber k.

The next step in the stability analysis, we shall examine stability diagrams of two special cases stated above (nonlinear diffusion equation and Schrödinger equation) of Ginzburg–Landau equation (4.13) in the dimensionless form as presented in (6.1).

We first discuss the case of incipient instability (nonlinear diffusion equation (5.1)), we consider the linear transition curve (6.2). In addition the nonlinear transition curves in the dimensionless approach (6.1) are

$$P_i = 0, \tag{6.4}$$

$$Q_i = 0. \tag{6.5}$$

In this case the stability diagrams are given in Figs. 9 and 10 are made for the variation of the electric field intensity E_0^2 versus the conductivity ratio $\sigma(= \sigma_1/\sigma_2)$ in range $\sigma < 1$. The linear and nonlinear transition curves are polynomials degree in E_0^2, divide the $E_0^2 - \sigma$ plane into stable and unstable regions. We may also observe that the curve

$$E_{cc}^2 = \left[T(1 - 4m^2 - 4k^2)(\sigma K_{2m}(2k)I'_{2m}(2k) - I_{2m}(2k)K'_{2m}(2k))\right] \times [2kI_{2m}(2k)K_{2m}(2k)(\varepsilon - 1)(\sigma - 1)]^{-1} \tag{6.6}$$

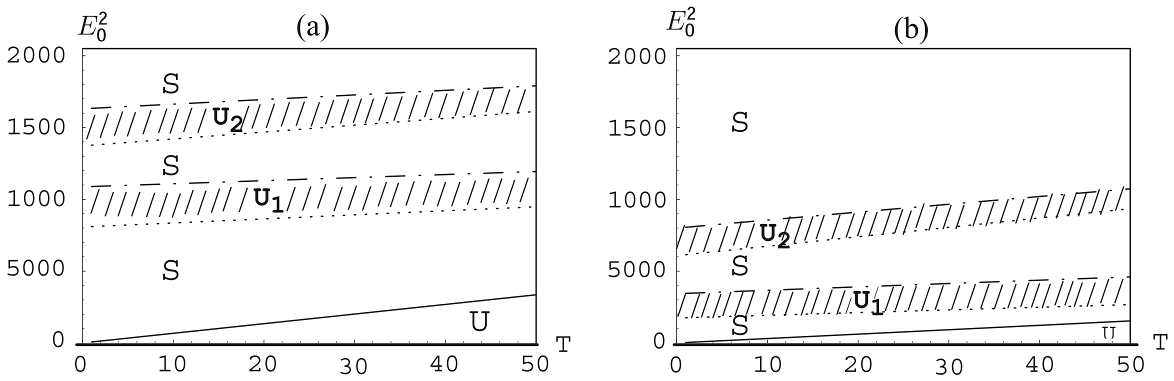

Fig. 8. Stability diagram in the $(E_0^2 - T)$-plane for nonlinear system having $m = 0$, $T = 1$, $\varepsilon = 0.5$ and $\rho = 1.5$. According to linear equations (6.2), (6.3) and nonlinear equations (4.15) and (4.16), where (a) refers to the case $k = 0.1$, (b) refers to the case $k = 0.2$. The continued line represents the linear curves (6.2) and (6.3). The dashed dotted line represents the nonlinear curve (4.15) and the dotted line represents the nonlinear curve (4.16).

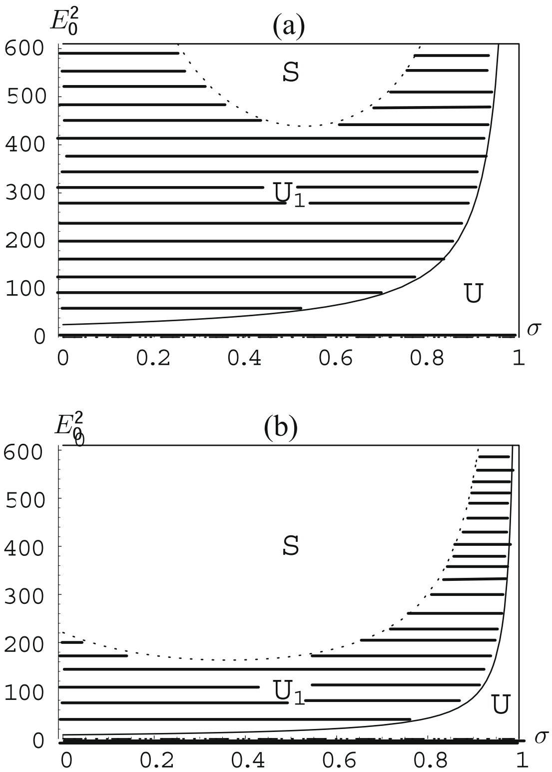

Fig. 9. Stability diagram in the $(E_0^2 - \sigma)$-plane for nonlinear system having $m = 0$, $T = 1$, $\varepsilon = 0.5$ and $\rho = 1.5$. According to linear equation (6.2), $P_i = 0$, $Q_i = 0$ and the second harmonic resonance (6.6), where (a) refers to the case $k = 0.2$, (b) refers to the case $k = 0.3$. The continued line represents the linear curve (6.2), and the dotted line represents the curve $Q_i = 0$. The second harmonic resonance curve and the curves of $P_i = 0$ lie in the negative part of E_0^2-axis.

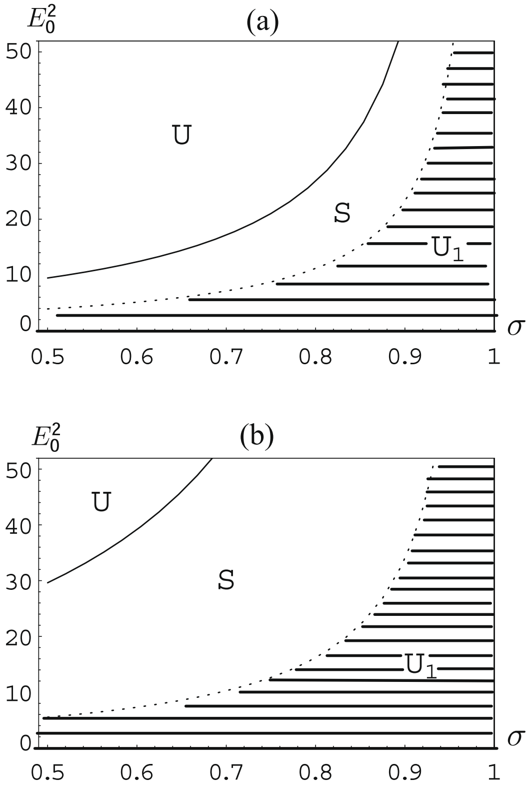

Fig. 10. Stability diagram for the same system as in Fig. 9, but with (a) refers to the case $k = 2$, (b) refers to the case $k = 5$.

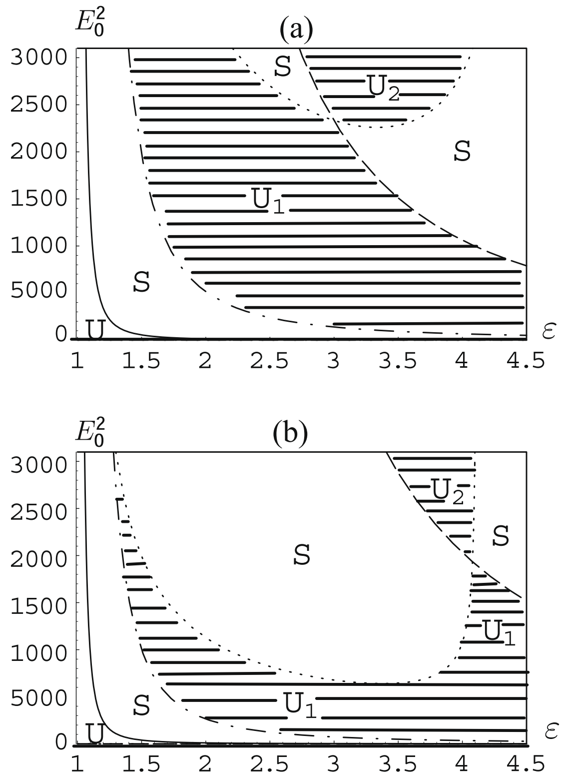

Fig. 11. Stability diagram in the $(E_0^2-\varepsilon)$-plane for nonlinear system having $m=0$, $\rho=1.5$, $T=1$ and $\sigma=0$. According to linear equation (6.3), $P_r=0$, $Q_r=0$ and the second harmonic resonance (6.7), where (a) refers to the case $k=0.2$, (b) refers to the case $k=0.25$. The continued line represents the linear curve (6.3). The dashed dotted line represents the curve (6.7), the dashed line represents $P_r=0$ and the dotted line represents $Q_r=0$.

is a third transition curve, because Q_i changes sign across this curve. The third curve represents the second harmonic resonance. It is interesting to explore the relation between the wavenumber and permittivity effects on the stability analysis, with the other parameters of the fluids held fixed at the values used for Figs. 9 and 10. These figure show how the tangential electric field is affected by increasing the wavenumber k which depends on the permittivity ratio ε. In Figs. 9 and 10, the continued curve represents the linear stability criterion (condition (6.2)), while the dotted ones indicate the nonlinear interaction parameter $Q_i=0$. The second harmonic resonance curve and the curves of the group velocity rate $P_i=0$ do not appear in these figures because it lie in the negative part of E_0^2-axis.

Fig. 9 displays the stability diagram in the $(E_0^2-\sigma)$-plane according to the case, $k<1$ and $\varepsilon<1$ for fixed values of $k=0.2$ and $k=0.3$ (Fig. 9a and b, respectively). Comparing Fig. 9a and b, we observe that the linear curve shift downwards, i.e., the stable region S increases with increasing k. This means that the stabilizing influence of the field, in the linear theory. It is also observed that the linear curve and the curve $Q_i=0$ produced a new region U_1. It is clear that the region U_1 decreases with the increase of k as shows in Fig. 9a and b. Therefore, the system is stable in view of the nonlinearity.

Fig. 10 represents the same system as considered in Fig. 9, but when $k>1$ and $\varepsilon>1$ for fixed values of $k=2$ (Fig. 10a) and $k=5$ (Fig. 10b). In Fig. 10 the new unstable region U_1 lie between the curves $\sigma=1$ and $Q_i=0$. Similarly as discussed in Fig. 9, the electric field has a stabilizing effect for both the linear and the nonlinear theory. We can say that, in the range $\sigma<1$, the electric field has a stabilizing effect, for the linear and the nonlinear theory, when $k<1$ and $\varepsilon<1$ or $k>1$ and $\varepsilon>1$. Otherwise, it has a destabilizing effect.

The second special case, when the electrical conductivities $\sigma_1=0$ and $\sigma_2=0$ are ignored. This case deals with the influence of the permittivity only (perfect dielectric fluids) corresponding to Schrödinger equation (5.3). If the solution of (5.3) is linearly perturbed, the perturbations are stable if condition $P_rQ_r>0$ is satisfied. On the other hand, if $P_rQ_r<0$ the system is unstable against modulation (i.e., the system does not oscillate about the steady state) and solitary waves propagate through the interface. Moreover, we see that the nonlinear Schrödinger equation (5.3) is invalid when $\Omega\to\infty$. This is the resonance case and it occurs when the denominator of Ω equals zero. Thus, if the above condition $P_rQ_r>0$ is satisfied, the finite deformation of the interface is stable and finite amplitude waves can propagate through the interface. However, many authors used to study the stability of a small modulation of the propagating wave by linear perturbing the solution of the Schrödinger equation [10,41]. Therefore, the linear transition curve is (6.3) and the nonlinear transition curves are given by (5.5) in the dimensionless form (6.1). Thus, the nonlinear transition curves are $P_r=0$, $Q_r=0$ and

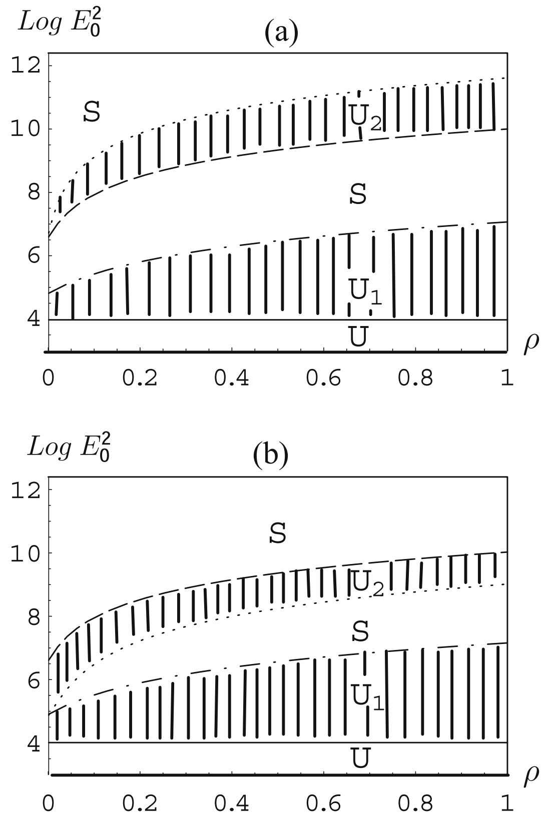

Fig. 12. Stability diagram in the $(LogE_0^2 - \rho)$-plane for the same system as in Fig. 11, but with $k = 0.2$ and $m = 0$, where (a) refers to the case $\varepsilon = 0.5$, (b) refers to the case $\varepsilon = 1.5$.

$$E_{cd}^2 = \left[T(1 - 4m^2 - 4k^2)\left(\varepsilon K_{2m}(2k)I'_{2m}(2k) - I_{2m}(2k)K'_{2m}(2k)\right)\right] \times (\varepsilon - 1)^{-2}(2kI_{2m}(2k)K_{2m}(2k))^{-1}, \tag{6.7}$$

which represents the second harmonic resonance curve corresponding to perfect dielectric fluids. Through the Figs. 11 and 12, we consider the continued line represents the linear curve representing the relation (6.3), which is assumed to divide the plane into a stable region S and an unstable region U. The dashed lines represents the curves of the group velocity rate $P_r = 0$. The dotted lines indicate the nonlinear interaction parameter $Q_r = 0$ and the dashed dotted lines refer to the second harmonic curve.

Fig. 11 is the stability diagram in the $(E_0^2 - \varepsilon)$-plane corresponding to the cases $k = 0.2$ and $k = 0.25$. We notice that the stable region S increases with k increasing because the linear curve shift downwards. The unstable regions (U_1, U_2) decrease when values of k increasing, where U_1 confined among the second harmonic curve, $P_r = 0$ and $Q_r = 0$ as shown in Fig. 11a, but lies between below the curve $Q_r = 0$ and upper the second harmonic curve as shown in Fig. 11b. while U_2 appears between the intersection the curve $Q_r = 0$ with the curve $P_r = 0$. This means that the electric field has a stabilizing effect on the linear system with increase the wavenumber and as does the nonlinearity.

Fig. 12 represents the variation of the stability charts in $(LogE_0^2 - \rho)$-plane, for $\varepsilon = 0.5$ and $\varepsilon = 1.5$ as in Fig. 12a and b, respectively. It is observed that the behaviour of the linear curve remains unchanged through the Fig. 12a and b. That is the linear theory, the permittivity ratio ε across the interface has no effect on the stability criterion of the system. So that the linear stability does not depend on the sign of $(\varepsilon_1 - \varepsilon_2)$. It is found that the nonlinear unstable regions (U_1, U_2) decrease as values of ε increases. This means that the linear system has not effect on the stability with respect to the permittivity ratio ε. While the inner and the outer dielectric plays a dual role in the nonlinear stability criterion in contrast with the linear analysis. Such effects, can be explained successfully in the nonlinear sense, as the linear analysis [1] unsuccessful to inform about them.

7. Conclusion

We have examined the linear and nonlinear electrohydrodynamic stability between two cylindrical inviscid fluids subjected to a tangential electric field. This investigation takes into account electric conductions in the fluids and the relaxation of electrical charges at the interface. Also, the influence of the surface tension is taken into account, while the gravitational forces are ignored. The perturbation technique is employed to third order. We have derived the general situation of the linear dispersion relation associated with the charge relaxation effect (overstability). We have studied the special situations where the fluids are the leaky dielectrics, as well as the situation where the fluids are perfect dielectrics. In the axisymmetric mode, found that the electric field can have either a stabilizing or a destabilizing effect depending on the ratios of conductivities and permittivities of the two fluids. While, the electric field always has a stabilizing effect for perfect dielectric fluids (does not depend on the permittivity ratio). Also we investigated the incipient static instability (characterized in the marginal state by $\omega = 0$) which was considered as a special case in our study. In this, we observed that the modes of cylinder have a stabilizing effect on the system.

Through the nonlinear theory, we obtained a nonlinear characteristic function. To this end, the multiple scales method in spatial and temporal developments is used to obtain uniformly valid expansions of the nonlinear characteristic function. By making use of the Fourier transform of this relation, we obtain a nonlinear dispersion relation (nonlinear Ginzburg–Landau equation). This general relation is useful, because it can be reduced to special relations for some cases by taking an approach for relevant physical parameters. For leaky dielectric fluids, as the nonlinear of incipient static instability (near the marginal state $\omega = 0$) is considerable, we obtained the nonlinear diffusion equation. Also, we found that the permittivities and conductivities ratios between the two fluids play a dual role in the nonlinear incipient static instability. For perfect dielectric fluids, two nonlinear Schrödinger equations are obtained. One of these equations is used to analyze the stability of the system, while the other is used to determine a nonlinear cutoff electric field separating stable and unstable disturbance. For perfect dielectric fluids, it is observed that the permittivity plays a dual role in the nonlinear stability. Such effects cannot be explained by linear analysis [1]. The analytical results are numerically confirmed. Stability diagrams are obtained for different sets of physical parameters. New instability regions existed in the linearly stable region due to nonlinear effects.

Some comparison with Elcoot [17] and the present work. Elcoot [17] showed that the radial electric field has a destabilizing effect, also for finite charge relaxation fluids has a destabilizing effect. But in the present work, it is found that the tangential electric field has a stabilizing effect, while for finite charge relaxation fluids the electric field can have either a stabilizing or a destabilizing influence depending on the conductivity and permittivity ratios of the two fluids.

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